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Completeness of multiseparable superintegrability on the complex 2-sphere

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Abstract. The possibility that Schrödinger's equation with a given potential can separate in more than one coordinate system is intimately connected with the notion of superintegrability. Here we demonstrate, for nondegenerate potentials, how to establish a complete list of such potentials that are superintegrable on the complex 2-sphere, using essentially algebraic means. We classify all such potentials that admit a pair of second-order constants of motion. Here 'nondegenerate' means that the potentials depend on four independent parameters. The method of proof generalizes to other spaces and dimensions. We show for the 2-sphere that all these superintegrable systems possess the remarkable property that they correspond to quadratic algebras, and we work out the detailed structure relations and their quantum analogues.

1. Introduction

It has long been known that Schrödinger's equation with certain special potentials can admit (multiplicative) separation of variables in more than one coordinate system. This is intimately related to the notion of superintegrability [1–4]. This subject has been studied by a number of authors, based on the use of the corresponding differential equations that are implied by the requirement of simultaneous separability [5–14]. Specifically, superintegrability means that for a Schrödinger equation in dimension N there exist $2N - 1$ functionally independent quantum mechanical observables (i.e. second-order[†] self-adjoint operators that commute with the Hamiltonian). There is an analogous concept of superintegrability for classical mechanical systems that relates to the corresponding additive separation of variables of the Hamilton–Jacobi equation. Also, if we do have simultaneous separability then the resulting constants of motion close quadratically under repeated application of the Poisson bracket [15, 16]. We also know that for spaces of constant curvature separable coordinate systems of the free motion are described by quadratic elements of the corresponding first-order symmetries [17]. Although concrete examples of superintegrable systems on constant-curvature spaces are easily at hand, a complete classification of all such systems has presented major difficulties. How can one be sure that all systems for free motion have been found? Once these are determined, how can one be sure that the most general additive potential term has been calculated?

[†] We restrict our definition to second-order symmetries because it is only these that could possibly be related to variable separation. A classification of superintegrable systems involving observables of arbitrary order remains open and it is not clear whether our method of integrability conditions would be tractable in the more general case.

In [18] we have solved the classification problem for all nondegenerate systems on two-dimensional complex Euclidean space. Here we solve the problem for the more difficult case of the two-dimensional complex sphere, including real spheres and hyperboloids as special cases. In particular, we classify all nondegenerate potentials that admit a pair of second-order constants of motion. Here ‘nondegenerate’ means that the potentials depend on four independent parameters. The requirement that a potential admit two constants of motion leads to two second-order partial differential equations obeyed by the potential, and the integrability conditions for these two simultaneous equations permit us to classify all possibilities. (We believe that this paper contains the first complete list of the possibilities, as well as a completeness proof. This is not a simple problem. For example [19] omits several of our cases.) The classification is greatly simplified by the equivalence of two potentials that are related by an action of the motion group $SO(3, C)$. We can prove that each nondegenerate potential is associated with a pair of constants of motion in the classical case, and a pair of symmetry operators in the quantum case, that, remarkably, generate a quadratic algebra. Furthermore, we verify that there is a one-to-one correspondence between superintegrable systems and free-field symmetry operators that generate quadratic algebras. Finally, we demonstrate explicitly that second-order superintegrability implies multiseparability, i.e. separability in more than one coordinate system.

2. Superintegrability on the complex sphere

We follow the approach of [18] and start by computing the possible second-order constants of motion for a Hamilton–Jacobi equation, with potential, on the complex sphere. (Inevitably, the only potentials that are candidates for superintegrability are those which are separable in more than one coordinate system on the two-dimensional complex sphere S_{2C} .) Here one considers the generators of the corresponding complex rotation group

$$J_1 = yp_z - zp_y \quad J_2 = zp_x - xp_z \quad J_3 = xp_y - yp_x. \quad (1)$$

The Hamilton–Jacobi equation is

$$H' = J_1^2 + J_2^2 + J_3^2 + V'(x, y, z) = E' \quad (2)$$

where $x^2 + y^2 + z^2 = 1$. To start with, it is convenient to make use of the natural embedding of the complex sphere in complex Euclidean 3-space. Consider the Hamiltonian

$$H = p_x^2 + p_y^2 + p_z^2 + V(x, y, z) \quad (3)$$

in E_{3C} , where $V = r^{-2}V'$ with $r^2 = x^2 + y^2 + z^2$. Then

$$H = r^{-2}(xp_x + yp_y + zp_z)^2 + r^{-2}H'$$

and we can identify the constants of motion L for H' with constants of motion for H such that

$$\{L, xp_x + yp_y + zp_z\} = \{L, r^2\} = 0. \quad (4)$$

Here of course

$$\{f, g\} = \sum_{j=1}^3 \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} \right) \quad (x_1, x_2, x_3) = (x, y, z). \quad (5)$$

We now determine the conditions that the function

$$L = \sum_{j,k=1}^3 a^{jk}(x, y) p_j p_k + W(x, y, z) = \ell + W \quad a^{jk} = a^{kj} \quad (6)$$

must satisfy to be a constant of motion for H' . The conditions (4) imply

$$\sum_j x_j \partial_{x_j} W = 0. \tag{7}$$

Furthermore, these conditions and the requirement $\{H, L\} = 0$ imply that the quadratic terms in L are expressible as a second-order polynomial in the J_k ,

$$\ell = AJ_3^2 + BJ_1^2 + CJ_2^2 + DJ_3J_1 + EJ_3J_2 + FJ_1J_2 \tag{8}$$

and that

$$\partial_{x_j} W = \sum_{k=1}^3 a^{jk} \partial_{x_k} V. \tag{9}$$

The integrability conditions

$$\partial_{x_k}(\partial_{x_j} W) = \partial_{x_j}(\partial_{x_k} W) \quad k \neq j$$

are

$$\begin{aligned} &(-2Axy + Dyz + Exz - Fz^2)(V_{yy} - V_{xx}) + (-2Cxz - Dy^2 + Eyx + Fyz)V_{yz} \\ &\quad + 2(A[y^2 - x^2] + [C - B]z^2 + Dxz - Eyz)V_{xy} \\ &\quad - (-2Byz + Dxy - Ex^2 + Fzx)V_{xz} \\ &= (-6Ay + 3Ez)V_x + (6Ax - 3Dz)V_y + (3Dy - 3Ex)V_z \\ &(-2Cxz - Dy^2 + Eyx + Fyz)(V_{zz} - V_{xx}) + (-2Axy + Dyz + Exz - Fz^2)V_{yz} \\ &\quad + 2(C[z^2 - x^2] + [A - B]y^2 - Eyz + Fxy)V_{zx} \\ &\quad - (-2Byz + Dxy - Ex^2 + Fzx)V_{yx} \\ &= (-6Cz + 3Ey)V_x + (-3Ex + 3Fz)V_y + (6Cx - 3Fy)V_z \\ &(-2Byz + Dxy - Ex^2 + Fzx)(V_{zz} - V_{yy}) + (-2Axy + Dyz + Exz - Fz^2)V_{xz} \\ &\quad + 2([A - C]x^2 + B[z^2 - y^2] - Dxz + Fxy)V_{zy} \\ &\quad - (-2Cxz - Dy^2 + Eyx + Fyz)V_{xy} \\ &= (-3Dy + 3Fz)V_x + (-6Bz + 3Dx)V_y + (6By - 3Fx)V_z. \end{aligned} \tag{10}$$

The homogeneity requirement on the embedded potential can be expressed as

$$xV_x + yV_y + zV_z = -2V$$

and this leads to the additional second-order conditions

$$\begin{aligned} xV_{xx} + yV_{xy} + zV_{xz} &= -3V_x \\ xV_{xy} + yV_{yy} + zV_{yz} &= -3V_y \\ xV_{xz} + yV_{zy} + zV_{zz} &= -3V_z. \end{aligned} \tag{11}$$

Note that here the ‘trivial’ solution for all of these equations is $V = c/r^2$.

One way to attack the problem of finding all superintegrable potentials on the sphere is to classify all potentials V that admit two functionally independent constants of motion and are ‘nondegenerate’ in the sense that they depend on four arbitrary constants, one of which can be considered to be the trivial constant c . The potential must satisfy two sets of equations of the form (10) and the conditions (11). These nine equations, not all independent, enable us to solve for the second derivatives $V_{xx}, V_{yy}, V_{zz}, V_{xz}, V_{yz}$ as linear combinations of the derivatives V_{xy}, V_x, V_y, V_z . Then all higher derivatives can be expressed in terms of those four, and integrability conditions obeyed by the higher derivatives imply linear relations between the four derivatives. *Nondegeneracy* of the potential means that at a nonsingular point (x_0, y_0, z_0) on the sphere we can prescribe the values of V_{xy}, V_x, V_y, V_z arbitrarily. Thus the coefficients of all linear relations between these derivatives must vanish identically for nondegenerate potentials.

Similarly, the linear relations between the terms in $V_{xx}, V_{yy}, V_{zz}, V_{xz}, V_{yz}$ on the left-hand sides of the original nine equations imply the same linear relations between V_{xy}, V_x, V_y, V_z , and these must also vanish identically. This approach to superintegrability on the sphere will prove useful in a forthcoming paper, where we study superintegrability in 3-space. However, we will adopt a simpler method for the remainder of this paper.

A second way to carry out the analysis (and the one that we shall follow) is directly in terms of coordinates x, y, z on the 2-sphere where $x^2 + y^2 + z^2 = 1$. We shall take x, y as independent variables and set $z = \pm\sqrt{1 - x^2 - y^2}$ with the sign depending on whether we are on the upper or lower hemisphere of S_{2C} . (Thus, we are adopting coordinates that correspond to the orthogonal projection of the sphere onto the tangent space at the north pole.) In some formulae we will adopt the convention $(x, y, z) = (y_1, y_2, y_3)$. In these coordinates the generators of the complex rotation group are

$$J_1 = -y_3 p_2 \quad J_2 = y_3 p_1 \quad J_3 = y_1 p_2 - y_2 p_1. \quad (12)$$

The Hamilton–Jacobi equation is

$$H' = (1 - y_1^2)p_1^2 - 2y_1 y_2 p_1 p_2 + (1 - y_2^2)p_2^2 + V(y_1, y_2) = E'. \quad (13)$$

Now the Poisson bracket of functions $f(y_1, y_2, p_1, p_2), g(y_1, y_2, p_1, p_2)$ is

$$\{f, g\} = \sum_{j=1}^2 \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial p_j} \right) \quad (y_1, y_2) = (x, y). \quad (14)$$

We next determine the conditions that the function

$$L = \sum_{j,k=1}^2 a^{jk}(y_1, y_2) p_j p_k + W(y_1, y_2) = \ell + W \quad a^{jk} = a^{kj} \quad (15)$$

must satisfy to be a constant of motion for H' . It is straightforward to show that the requirement $\{H', L\} = 0$ implies that the quadratic terms in L are expressible as a second-order polynomial in the J_k ,

$$\ell = \alpha_1 J_3^2 + \alpha_2 J_1^2 + \alpha_3 J_2^2 + \alpha_4 J_3 J_1 + \alpha_5 J_3 J_2 + \alpha_6 J_1 J_2 \quad (16)$$

and that (with $\partial_{y_j} W \equiv W_j$, etc)

$$W_1 = \left(\frac{(1 - y_2^2)a}{y_3^2} + \frac{y_1 y_2 b}{2y_3^2} \right) V_1 + \left(\frac{(1 - y_2^2)b}{2y_3^2} + \frac{y_1 y_2 c}{y_3^2} \right) V_2 \quad (17)$$

$$W_2 = \left(\frac{y_1 y_2 a}{y_3^2} + \frac{(1 - y_1^2)b}{2y_3^2} \right) V_1 + \left(\frac{y_1 y_2 b}{2y_3^2} + \frac{(1 - y_1^2)c}{y_3^2} \right) V_2 \quad (18)$$

where

$$\begin{aligned} a(y_1, y_2) &= (\alpha_1 - \alpha_3)y_2^2 + \alpha_3(1 - y_1^2) - \alpha_5 y_2 y_3 \\ b(y_1, y_2) &= -2\alpha_1 y_1 y_2 + \alpha_4 y_2 y_3 + \alpha_5 y_1 y_3 - \alpha_6(1 - y_1^2 - y_2^2) \\ c(y_1, y_2) &= (\alpha_1 - \alpha_2)y_1^2 + \alpha_2(1 - y_2^2) - \alpha_4 y_1 y_3. \end{aligned} \quad (19)$$

The integrability conditions

$$\partial_x(\partial_y W) = \partial_y(\partial_x W)$$

are

$$\begin{aligned} &[2(\alpha_3 - \alpha_1)xyz + \alpha_4 y(1 - x^2) + \alpha_5 x(1 - x^2 - 2y^2) + \alpha_6 z(1 - x^2)] \\ &\times \left(V_{xx} - \frac{1 - y^2}{xy} V_{xy} \right) \end{aligned}$$

$$\begin{aligned}
 & +[2(\alpha_1 - \alpha_2)xyz - \alpha_4y(1 - 2x^2 - y^2) - \alpha_5x(1 - y^2) - \alpha_6z(1 - y^2)] \\
 & \times \left(V_{yy} - \frac{1 - x^2}{xy} V_{xy} \right) \\
 = & (6[\alpha_1 - \alpha_3]yz + 3\alpha_4xy - 3\alpha_5(1 - x^2 - 2y^2) + 3\alpha_6xz)V_x \\
 & + (6[\alpha_2 - \alpha_1]xz + 3\alpha_4(1 - 2x^2 - y^2) - 3\alpha_5xy - 3\alpha_6yz)V_y. \tag{20}
 \end{aligned}$$

We denote the V solution space of this equation by

$$[\alpha_1, \dots, \alpha_6]. \tag{21}$$

Let us now return to our assumption that the Hamilton–Jacobi equation admits two constants of motion:

$$L_h = \sum_{j,k=1}^2 a_{(h)}^{jk} p_k p_j + W_{(h)} \quad h = 1, 2.$$

These two operators together with H' are assumed to be functionally independent. The constant of motion L_1 leads to the condition (21) on the potential V , whereas L_2 leads to the second condition

$$[\beta_1, \dots, \beta_6]. \tag{22}$$

The potential must lie in the intersection of the solution spaces (21), (22) for these two conditions. It follows that the equations

$$V_{xx} - \frac{1 - y^2}{xy} V_{xy} = AV_x + BV_y \quad V_{yy} - \frac{1 - x^2}{xy} V_{xy} = CV_x + DV_y \tag{23}$$

must hold, where

$$\begin{aligned}
 A\mathcal{E} = & -12H_{12}xy^2z^2 + 6H_{14}y^2z(y^2 - 1 + x^2) + 12H_{13}xy^2z^2 + 6H_{16}yz^2(y - x^2 - 1) \\
 & - 6H_{15}xyz(x^2 + y^2) + 6H_{25}xyz(2y^2 + x^2 - 1) + 6H_{26}x^2yz^2 \\
 & - 12H_{23}xy^2z^2 + 6H_{24}x^2y^2z + 6H_{34}y^2z(1 - 2x^2 - y^2) \\
 & + 6H_{36}yz^2(1 - y^2) + 6H_{35}xyz(1 - y^2) \\
 & + 3H_{45}y(2x^2 - 1 - 2y^4 - 4x^2y^2 + 3y^2 - 2x^4) \\
 & - 6H_{46}x^3yz + 3H_{56}z(2y^4 - 3y^2 + 1) \\
 B\mathcal{E} = & 6H_{15}x^2z + 12H_{14}xyz(1 - y^2 - 2x^2) + 6H_{16}xz^2 - 6H_{26}xz^2 \\
 & + 12H_{24}xyz(2x^2 + y^2 - 1) - 6H_{25}x^2z + 3H_{46}z(1 - 2x^2 - y^2)(1 - 2y^2) \\
 & + 3H_{45}x(1 - 2x^2 - y^2)(1 - 2y^2) \\
 2C\mathcal{E} = & -6H_{16}yz^2 - 6H_{14}y^2z + 6H_{36}yz^2 + 6H_{34}y^2z \\
 & + 3H_{45}y(2y^2 + x^2 - 1) + 3H_{56}z(1 - 2y^2 - x^2) \\
 2D\mathcal{E} = & -12H_{12}x^2yz^2 + 6H_{16}xz^2(1 - x^2 + y^2) + 6H_{15}x^2z(1 - y^2 - x^2) \\
 & + 12H_{13}x^2yz^2 + 6H_{14}xyz(2 - 3x^2 - y^2) + 6H_{24}xyz(x^2 - 1) \\
 & - 6H_{26}xz^2(1 - x^2) + 6H_{25}x^2z(2y^2 + x^2 - 1) - 12H_{23}x^2yz^2 \\
 & + 6H_{34}xyz(2x^2 + y^2 - 1) - 6H_{36}xy^2z^2 - 6H_{35}x^2y^2z \\
 & + 3H_{45}x(1 - 4y^2 + 6x^2y^2 - 3x^2 + 2x^4 + 2y^4) \\
 & + 3H_{46}z(1 - x^2)(1 - 2x^2 - 2y^2) + 6H_{56}xy^3z \\
 \mathcal{E} = & -4H_{13}x^2y^2(1 - x^2 - y^2) + 2H_{16}(1 - x^2)(x^2 - y^2)xy + 2H_{15}x^2yz(y^2 + x^2) \\
 & + 4H_{12}x^2y^2(1 - x^2 - y^2) - 2H_{14}xy^2z(x^2 + y^2)
 \end{aligned} \tag{24}$$

$$\begin{aligned}
& -2H_{26}xy(1-x^2-y^2)(x^2-1) \\
& +2H_{24}xy^2z(1-x^2) - 2H_{25}x^2yz(2y+x^2-1) \\
& +4H_{23}x^2y^2(1-x^2-y^2) \\
& +2H_{36}xy(1-x^2-y^2)(y^2-1) - 2H_{34}xy^2z(1-2x^2-y^2) \\
& +2H_{35}x^2yz(y^2-1) \\
& -2H_{45}xy(y^2-y^4-2x^2y^2-x^4+x^2) \\
& -2H_{46}x^2yz(1-x^2) - 2H_{56}xy^2z(y^2-1)
\end{aligned}$$

and $H_{k\ell} = -H_{\ell k} = \alpha_k\beta_\ell - \alpha_\ell\beta_k$.

Differentiating each of equations (23) with respect to x and y , we obtain four equations for the four third-derivatives of V , expressed in terms of V_x, V_y, V_{xy} :

$$\begin{aligned}
V_{xxx} = & \frac{1}{xy(-1+y^2+x^2)}((xy^3-xy+yx^3)B_x \\
& +(yx-2y^3x)\mathcal{E}_x + (yx^3-yx+y^3x)AB \\
& +(y^2x^2-y^4x^2)B_y + (yx+y^5x)CB \\
& +y^5x\mathcal{E}_x - 2y^3xCB + (y^2x^2-y^4x^2)B\mathcal{E})V_y \\
& + \frac{1}{yx(-1+y^2+x^2)}((x^3y-xy+xy^3)A_x + (y^2x^2-y^4x^2)A_y \\
& +(yx+y^5x-2y^3x)C_x + (yx+y^5x-2y^3x)CA + (y^2x^2-y^4x^2)BC \\
& +(y^3x+yx^3-yx)A^2)V_x \\
& + \frac{1}{yx(-1+y^2+x^2)}((2y^2+x^2-y^4-y^4x^2-1)A \\
& +(3y^4-y^6-3y^2)C + y^3x^3B + yx\mathcal{E} \\
& -3x-2y^3x\mathcal{E} + 3xy^2 + y^5x\mathcal{E})V_{xy}
\end{aligned} \tag{25}$$

$$\begin{aligned}
V_{xyy} = & \frac{-1}{x(-1+y^2+x^2)}(-x^3y^2BE + y^3x^2\mathcal{E}_x - x^3y^2B_y \\
& -x^2yCB - x^2y\mathcal{E}_x + y^3x^2CB)V_y \\
& - \frac{1}{x(-1+y^2+x^2)}(-x^3y^2A_y + y^3x^2C_x - x^2yCA + y^3x^2CA \\
& -x^2yC_x - x^3y^2BC)V_x \\
& - \frac{1}{x(-1+y^2+x^2)}(-x^2yB - x^3y^2A - y^2 - x^2y\mathcal{E} + 1 + y^3x^2\mathcal{E} \\
& -xy^4C - xC + 2x^2 + x^4yB + 2xy^2C)V_{xy}
\end{aligned} \tag{26}$$

$$\begin{aligned}
V_{xyy} = & \frac{1}{y(-1+y^2+x^2)}(-x^3y^2B\mathcal{E} + (xy^2-x^3y^2)B_y + y^2x\mathcal{E} \\
& +y^3x^2\mathcal{E}_x + y^3x^2CB)V_y \\
& + \frac{1}{y(-1+y^2+x^2)}((xy^2-x^3y^2)A_y + (y^2x-x^3y^2)BC \\
& +y^3x^2CA + y^3x^2C_x)V_x \\
& + \frac{1}{y(-1+y^2+x^2)}((xy^2-x^3y^2)A + (x^4y-2x^2y+y)B \\
& +x^2+y^3x^2\mathcal{E} - (xy^4+xy^2)C(x,y) - 2y^2-1)V_{xy}
\end{aligned} \tag{27}$$

$$\begin{aligned}
 V_{yyy} = & \frac{1}{yx(-1+y^2+x^2)}((x^3y+xy^3-xy)\mathcal{E}_y+(y^2x^2-x^4y^2)\mathcal{E}_x-x^4y^2CB \\
 & +yxB\mathcal{E}+x^5yB_y+yx^3\mathcal{E}^2+(xy-2x^3y)B_y-2x^3yB\mathcal{E} \\
 & +y^2x^2BC+x^5yB\mathcal{E}+(y^3x-yx)\mathcal{E}^2)V_y \\
 & +\frac{1}{yx(-1+y^2+x^2)}((y^2x^2-x^4y^2)C_x+(xy+x^5y-2x^3y)A_y \\
 & +(x^3y-xy+xy^3)C_y+(y^3x-yx+x^3y)\mathcal{E}C+(y^2x^2-x^4y^2)CA \\
 & +(xy-2x^3y+x^5y)BC)V_x \\
 & +\frac{1}{yx(-1+y^2+x^2)}((y^2-x^4y^2+2x^2-x^4-1)\mathcal{E} \\
 & +(x^5y+xy-2x^3y)A+x^3y^3C+(3x^4-x^6-3x^2+1)B \\
 & -3y+3x^2y)V_{xy}.
 \end{aligned} \tag{28}$$

Thus if the potential V is subject to the two conditions (21), (22), then V can depend on at most three parameters, in addition to a trivial additive constant. We can choose these parameters to be $V_x(x_0, y_0)$, $V_y(x_0, y_0)$, $V_{xy}(x_0, y_0)$ for any fixed regular point (x_0, y_0) . Then $V_{xx}(x_0, y_0)$, $V_{yy}(x_0, y_0)$ and all higher derivatives can be computed in terms of V_x, V_y, V_{xy} by successive differentiation and utilization of relations (23).

We require that our potential be *nondegenerate*, i.e. that it depend on three arbitrary parameters. Then, the three conditions $\partial_y V_{xxx} = \partial_x V_{xxy}$, $\partial_y V_{xxy} = \partial_x V_{xyy}$ and $\partial_y V_{xyy} = \partial_x V_{yyy}$ for the fourth partial derivatives lead to nine integrability conditions, since we can equate the coefficients of V_x, V_y and V_{xy} in each of these identities. (Otherwise V would necessarily depend on fewer than three arbitrary parameters.)

Note that if we have another constant of motion L_3 associated with a nondegenerate potential, then L_3 must be a linear combination of $H' = J_1^2 + J_2^2 + J_3^2, L_1, L_2$. Indeed, if L_3 is not a linear combination of the basis functions, then the potential V must satisfy an equation (10) that is linearly independent of the equations associated with L_1, L_2 . This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

The integrability conditions are only guaranteed to be necessary conditions for the existence of a three-parameter potential. However, we shall find that they are in fact sufficient. Each of the nine conditions can be expressed as a polynomial identity in the variables x, y whose coefficients are homogeneous polynomials in the coefficients H_{ij} . Since these relations must hold identically in x, y we can equate to zero each of the components in the polynomial expansion. The resulting expressions are lengthy; we used the symbol manipulation program MAPLE to compute them. Even then they would have been very cumbersome to solve if we had not been able to simplify the computation further by taking advantage of $SO(3, C)$ equivalence.

We will now use the nine conditions to classify the possible potentials V and the corresponding constants of motion L_1, L_2 . For this we note that it is only the three-dimensional subspace spanned by H, L_1, L_2 that matters; we can choose any basis for this subspace. Hence we can replace the conditions (21), (22) by linear combinations of themselves without changing the potential. Moreover, to further simplify the results we note that we can always subject the coordinates (x, y) , and L_1, L_2 to a simultaneous complex rotation motion, i.e. we regard all translated and rotated potentials as members of the same equivalence class.

We will consider two superintegrable systems on the complex sphere as the same if one system can be transformed to the other via an action of the complex orthogonal group $SO(3, C)$. One can identify the adjoint action of $SO(3, C)$ on the second-order elements in the enveloping

algebra of $so(3, C)$ with the action via similarity transformation of this group on the space of 3×3 complex symmetric matrices. (Here, $J \cdot J$ corresponds to the identity matrix.) A straightforward computation shows that this actions divides the second-order elements into orbits (or equivalence classes). A representative from each orbit class is given by

$$J_3^2 \quad (\text{spherical}) \quad (29)$$

$$(J_1 + iJ_2)^2 \quad (\text{horospherical}) \quad (30)$$

$$J_3(J_1 + iJ_2) \quad (\text{degenerate elliptic of type 2}) \quad (31)$$

$$J_1^2 + r^2 J_2^2 \quad |r^2| \leq 1 \quad r^2 \neq 0, 1 \quad (\text{elliptic}) \quad (32)$$

$$(J_1 + iJ_2)^2 + sJ_3^2 \quad s \neq 0 \quad (\text{degenerate elliptic of type 1}) \quad (33)$$

$$J \cdot J = J_1^2 + J_2^2 + J_3^2. \quad (34)$$

Here, $J \cdot J$ is invariant under the group action, and we can add arbitrary multiples of $J \cdot J$ to any of these operators without changing the orbit class. See the appendix for a description of the separable coordinates corresponding to these operators.

Our strategy to classify the three-dimensional subspaces of operators corresponding to maximal parameter-dependent potentials is as follows. We choose ℓ_1 from one of the orbit classes (29)–(33), where $L_1 = \ell_1 + W_1$. We first take ℓ_2 , ($L_2 = \ell_2 + W_2$) as a general operator $[\beta_1, \dots, \beta_6]$. However, we can then simplify ℓ_2 by adding arbitrary multiples of ℓ_1 and $J \cdot J$ to it, by multiplying ℓ_2 by any nonzero complex number and by applying a complex rotation to ℓ_2 that leaves ℓ_1 invariant. Note that $J \cdot J$ is invariant under all complex rotations. Finally we apply the nine integrability conditions to L_1 and the simplified L_2 , to determine those choices of L_2 that admit the nondegenerate potentials. We do this for each of the five choices of L_1 , and then eliminate any duplications to obtain the final list of all nondegenerate potentials. (For more details of the steps in this process, see [18] where the same computation was carried out for two-dimensional complex Euclidean space.)

Suppose $\ell_1 = J_3^2$. This operator is invariant under an arbitrary complex rotation about the z -axis. Such a rotation will leave β_1 , $I_1 = \beta_2 + \beta_3$, $I_2 = \beta_2\beta_3 - \beta_6^2/4$ and $I_3 = \beta_4^2 + \beta_5^2$ invariant. If we can rotate such that the transformed $\beta_6 = 0$ then we can achieve the form $[0, 1, 0, \beta_4, \beta_5, 0]$. The integrability conditions require $\beta_4 = \beta_5 = 0$.

Case 1a.

$$[1, 0, 0, 0, 0, 0] \quad [0, 1, 0, 0, 0, 0]. \quad (35)$$

Here,

$$L_1 = J_3^2 + W^{(1)} \quad L_2 = J_1^2 + W^{(2)} \quad (36)$$

$$V(x) = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}. \quad (37)$$

This potential allows separation in spherical and elliptic coordinates. (Note: due to expressions (17), (18), it is always a straightforward integration to compute the terms $W^{(j)}$, and we will not list these explicitly. Furthermore, once the constants of motion for a superintegrable system are known, it is relatively straightforward to determine the possible separable coordinate systems associated with this system. This is due to the fact that the constants of motion for the separable coordinate systems are already known (see the appendix). The only complication is that we may have to apply an $SO(3, C)$ transformation to the standardized constant of motion for the separable system to obtain the corresponding symmetry of the superintegrable system. Here, we simply list the separable systems associated with each superintegrable system, and provide details only in the cases where shifted coordinates occur.)

If $I_1 = I_2 = 0$ but not all of $\beta_2, \beta_3, \beta_6$ are zero, then we can rotate to achieve $[0, 1, -1, \beta_4, \beta_5, 2i]$. Again integrability conditions require $\beta_4 = \beta_5 = 0$.

Case 1b.

$$[1, 0, 0, 0, 0, 0] \quad [0, 1, -1, 0, 0, 2i]. \tag{38}$$

Here,

$$L_1 = J_3^2 + W^{(1)} \quad L_2 = (J_1 + iJ_2)^2 + W^{(2)} \tag{39}$$

$$V(x) = \frac{\alpha}{z^2} + \frac{\beta}{(x + iy)^2} + \frac{\gamma(x - iy)}{(x + iy)^3}. \tag{40}$$

This potential allows separation in spherical, horospherical and degenerate elliptical coordinates of type 1.

If $\beta_2 = \beta_3 = \beta_6 = 0, I_3 = 0$, we can achieve $\beta_5 = -i\beta_4$, which satisfies the integrability conditions.

Case 1c.

$$[1, 0, 0, 0, 0, 0] \quad [0, 0, 0, 1, -i, 0]. \tag{41}$$

Here,

$$L_1 = J_3^2 + W^{(1)} \quad L_2 = J_3(J_1 + iJ_2) + W^{(2)} \tag{42}$$

$$V(x) = \frac{\alpha}{(x + iy)^2} + \frac{\beta z}{\sqrt{x^2 + y^2}} + \frac{i\gamma}{\sqrt{(x - iy)(x + iy)^2}}. \tag{43}$$

Separability is possible in spherical coordinates and degenerate elliptic coordinates of type 2.

If $I_3 \neq 0$ we have the following.

Case 1d.

$$[1, 0, 0, 0, 0, 0] \quad [0, 0, 0, 1, 0, 0]. \tag{44}$$

Here,

$$L_1 = J_3^2 + W^{(1)} \quad L_2 = J_3 J_1 + W^{(2)} \tag{45}$$

$$V(x) = \frac{\alpha z}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{x^2 + y^2}} \left[\frac{\beta}{\sqrt{x^2 + y^2} + x} + \frac{\gamma}{\sqrt{x^2 + y^2} - x} \right]. \tag{46}$$

Separation of variables is possible in spherical coordinates and rotated elliptic coordinates, $\{R_1, R_2\}$. A suitable choice of the latter is

$$z_1 = \frac{i(A_+^2 + A_-^2)(R_1 R_2 + 1) + 2A_+ A_- (R_1 + R_2)}{(A_+^2 - A_-^2)\sqrt{R_1 R_2}} \quad z_3 = -\frac{1}{2} \frac{R_1 R_2 + 1}{\sqrt{R_1 R_2}}$$

$$z_2 = \frac{\sqrt{(A_+ R_1 + A_-)(A_- R_1 + A_+)(A_+ R_2 + A_-)(A_- R_2 + A_+)}}{(A_+^2 - A_-^2)\sqrt{R_1 R_2}}$$

with the associated operator $(A_-^2 - A_+^2)J_1 J_3 - i(A_+^2 + A_-^2)J_3^2$.

Now suppose $\ell_1 = J_1^2 + r^2 J_2^2$, corresponding to (32). In this case there is no simplification possible by rotation and we must apply the integrability conditions for general $[0, 0, \beta_3, \beta_4, \beta_5, \beta_6]$. We find that the integrability conditions are satisfied if and only if $\beta_3 = \pm ir, \beta_4 = \beta_5 = 0$ and $\beta_6 = 1$.

Case 2.

$$[0, 1, r^2, 0, 0, 0] \quad [0, 0, ir, 0, 0, 1]. \tag{47}$$

Here,

$$L_1 = (c^2 - 1)^2 J_3^2 - 4c^2 J_1^2 + W^{(1)} \quad L_2 = c^2 J_3^2 - (J_1 - iJ_2)^2 + W^{(2)}$$

$$V(x) = \frac{\alpha(z_+ + c^2 z_-)}{\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3^2}} + \frac{\beta(z_+ - c^2 z_-)(z_+ z_- + z_3^2)}{z_3^2 \sqrt{(c^2 z_- - z_+)^2 - 4c^2 z_3^2}} + \frac{\gamma z_+ z_-}{z_3^2} \tag{48}$$

$z_{\pm} = x \pm iy$, $z_3 = \sqrt{1 - x^2 - y^2}$ and $c^2 = (1 + r)/(1 - r)$. Separation of variables is possible in elliptical and degenerate elliptic coordinates of type 1.

Next suppose $\ell_1 = (J_1 - iJ_2)^2$. Then, eliminating case 1b above, the only possibilities for L_2 are operators of the form $[\beta_1, 1, 0, \beta_4, \beta_5, 0]$, or $[\beta_1, 0, 0, 1, 0, 0]$, or $[\beta_1, 0, 0, 1, -i, 0]$, and only the last of these with $\beta_1 = 0$ satisfies the integrability conditions.

Case 3.

$$[0, 1, -1, 0, 0, -2i] \quad [0, 0, 0, 1, -i, 0] \tag{49}$$

where

$$L_1 = (J_1 - iJ_2)^2 + W^{(1)} \quad L_2 = J_3(J_1 - iJ_2) + W^{(2)}$$

$$V(x) = \frac{\alpha}{(x + iy)^2} + \frac{\beta z}{(x + iy)^3} + \frac{\gamma(1 - 4z^2)}{(x + iy)^4}. \tag{50}$$

Separation of variables is possible in horospherical coordinates and degenerate elliptic coordinates of type 2.

One can verify from the integrability conditions that orbit (33) does not occur for any nondegenerate potential, other than the cases already listed. This completes the classification of these potentials. The results are summarized in table 1.

For a general choice of operators L_1, L_2 it is not the case that $R^2 = \{L_1, L_2\}^2$ is a polynomial in L_0, L_1, L_2 , i.e. there is no quadratic algebra structure. However, we can demonstrate that there *is* a quadratic algebra associated with each nondegenerate potential above. Because we are working in two dimensions there can only be three functionally independent constants at most. Consequently all Poisson brackets must be functionally dependent on $H = L_0, L_1$ and L_2 . We want to show that in fact $R^2 = \{L_1, L_2\}^2 = F(L_0, L_1, L_2)$ is a polynomial in these variables. First, we can verify that this is true when the potential is turned off, i.e. if we consider only the functions

$$\ell_h = \sum_{j,k=1}^2 a_{(h)}^{jk} p_k p_j \quad h = 1, 2 \quad \ell_0 = (1 - y_1^2) p_1^2 - 2y_1 y_2 p_1 p_2 + (1 - y_2^2) p_2^2$$

where $L_h = \ell_h + W^{(h)}$. Let $\mathcal{R} = \{\ell_1, \ell_2\}$. Then for each of the cases 1–3 listed above it is straightforward to check that $\mathcal{R}^2 = \mathcal{P}_3(\ell_0, \ell_1, \ell_2)$ where \mathcal{P}_3 is a homogeneous third-order polynomial in its arguments†. It follows that

$$R^2 = F(L_0, L_1, L_2) = \mathcal{P}_3(L_0, L_1, L_2) + F_4(s, L_0, L_1, L_2) \tag{51}$$

where F_4 is a fourth-, second- and zeroth-order polynomial in the momenta p_x, p_y , and $F_4(\mathbf{0}, L_0, L_1, L_2) = 0$. Here, the parameters in the potential are denoted by $s = (V_x^0, V_y^0, V_{yy}^0)$, evaluated at some fixed point (x_0, y_0) , and F_4 is a polynomial function of these parameters.

† Moreover, it is straightforward to verify that the cases corresponding to nondegenerate potentials are the *only* cases where \mathcal{P}_3 is a homogeneous third-order polynomial in its arguments. Thus the possible quadratic algebras generated by second-order elements in the Lie algebra of $SO(3, C)$ correspond one to one with nondegenerate potentials.

Table 1.

Potential V	Coordinate system
$V_1 = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}$	Spherical Elliptic
$V_2 = \frac{\alpha z}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{x^2 + y^2}} \times \left(\frac{\beta}{\sqrt{x^2 + y^2} + x} + \frac{\gamma}{\sqrt{x^2 + y^2} - x} \right)$	Spherical Elliptic II
$V_3 = \frac{\alpha}{(x + iy)^2} + \frac{\beta z}{(x + iy)^3} + \frac{\gamma(1 - 4z^2)}{(x + iy)^4}$	Horospherical Degenerate elliptic of type 2
$V_4 = \frac{\alpha}{z^2} + \frac{\beta}{(x + iy)^2} + \frac{\gamma(x - iy)}{(x + iy)^3}$	Spherical Horospherical Degenerate elliptic of type 1
$V_5 = \frac{\alpha}{(x + iy)^2} + \frac{\beta z}{\sqrt{x^2 + y^2}} + \frac{i\gamma\sqrt{x - iy}}{x^2 + y^2}$	Spherical Degenerate elliptic of type 2
$V_6 = \frac{\alpha(z_+ + c^2 z_-)}{\sqrt{(c^2 z_- - z_+)^2 - 4c^2 z^2}} + \frac{\beta(z_+ - c^2 z_-)(z_+ z_- + z^2)}{z^2 \sqrt{(c^2 z_- - z_+)^2 - 4c^2 z^2}} + \frac{\gamma z_+ z_-}{z^2}$ $z_{\pm} = x \pm iy \quad c^2 = (1 + r)/(1 - r)$	Elliptic Degenerate elliptic of type 1

From the definition of the Poisson bracket we have

$$\{\ell_1, \mathcal{R}\} = \frac{1}{2} \frac{\partial \mathcal{P}_3}{\partial \ell_2}(\ell_0, \ell_1, \ell_2)$$

$$\{\ell_2, \mathcal{R}\} = -\frac{1}{2} \frac{\partial \mathcal{P}_3}{\partial \ell_1}(\ell_0, \ell_1, \ell_2)$$

hence

$$\{L_1, R\} = \frac{1}{2} \frac{\partial \mathcal{P}_3}{\partial L_2}(L_0, L_1, L_2) + \frac{1}{2} \frac{\partial F_4}{\partial L_2}(\mathbf{s})$$

$$\{L_2, R\} = -\frac{1}{2} \frac{\partial \mathcal{P}_3}{\partial L_1}(L_0, L_1, L_2) - \frac{1}{2} \frac{\partial F_4}{\partial L_1}(\mathbf{s})$$

where the $\partial F_4/\partial L_h(\mathbf{s})$ have only terms of orders two and zero in the momenta. It follows that the $\partial F_4/\partial L_h(\mathbf{s})$ must be expressible as linear combinations of the L_h . This shows that the commutators $\{L_h, R\}$ can be expressed as polynomials in L_0, L_1, L_2 . It is then a simple matter to verify that F itself is a polynomial in L_0, L_1, L_2 .

We now list the quadratic algebra relations for each of the cases studied above. In view of relations

$$\{L_1, R\} = \frac{1}{2} \frac{\partial F}{\partial L_2} \quad \{L_2, R\} = -\frac{1}{2} \frac{\partial F}{\partial L_1}. \tag{52}$$

it is sufficient to give the relation $R^2 = F(L_0, L_1, L_2)$ for each case.

Case 1a. [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0]

$$R^2 = 16L_2L_1(H' - L_1 - L_2 - \alpha - \beta - \gamma) - 16[\alpha L_2^2 + \beta(H' - L_1 - L_2 - \alpha - \beta - \gamma)^2 + \gamma L_1^2] + 64\alpha\beta\gamma.$$

Case 1b. [1, 0, 0, 0, 0, 0], [0, 1, -1, 0, 0, 2i]

$$R^2 = -16L_2^2L_1 - 16\gamma H'^2 - 16\gamma L_1^2 - 16\beta H'L_2 - 32\gamma HL_1 - 16\beta L_2L_1 + 16\alpha\beta L_2 - 32\alpha\gamma L_1 + 32\alpha\gamma H' + 16\alpha\gamma(\gamma - \alpha).$$

Case 1c. [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, -i, 0]

$$R^2 = -4L_2^2L_1 - 4\alpha L_1^2 + 4\alpha H'L_1 + 2\gamma L_2L_1 + i\beta\gamma L_2 - \frac{\gamma^2}{4}H' + \beta^2\alpha - \frac{i\beta^2}{4}.$$

Case 1d. [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]

$$R^2 = -4L_1^3 - 4L_2^2L_1 + 4L_1^2H' - 2(\beta + \gamma)L_1^2 - 2(\beta + \gamma)L_2^2 + 4\alpha L_1L_2 + 4\beta\gamma L_1 + 4\alpha\beta L_2 - 4\beta\gamma H' + 2\beta\gamma(\beta + \gamma) - 2\beta\alpha^2.$$

Case 2. [0, 1, r^2 , 0, 0, 0], [0, 0, ir , 0, 0, 1]

$$R^2 = 32c^4(c^2 - 1)H'^3 - 16c^2(c^4 - 1)L_2^3 - 16c^2L_1^2L_2 + 32c^2(c^2 - 1)(2c^2 + 1)L_2^2H' + 16(2c^2 - 1)L_2^2L_1 - 16(c^2 - 1)(5c^2 + 1)c^2H'^2L_2 + 16c^4H'^2L_1 - 32c^2(2c^2 - 1)L_2L_1H' + 64\gamma(c^4 - 1)L_2^2 - 128\gamma c^2(c^2 - 1)H'L_2 - 64\gamma c^2L_2L_1 + 16c^2[(c^2(\alpha - \beta))^2 - (\alpha + \beta)^2]L_2 - 16c^4(\beta - \alpha)^2L_1 - 32c^4(\beta - \alpha)(c^2(\beta - \alpha) + \beta + \alpha)H' - 64c^4\gamma(\beta - \alpha)^2.$$

Case 3. [0, 1, -1, 0, 0, 2i], [0, 0, 0, 1, i, 0]

$$R^2 = 4L_1^3 + 16\gamma L_2^2 + 8\alpha L_1^2 + 16\gamma H'L_1 - 4\beta L_2L_1 - 4\beta\alpha L_2 - \beta^2H'.$$

3. Quantum superintegrability on the two-dimensional sphere

Here we give the analogous quantum algebras for superintegrable systems arising from the potentials we have already computed. The main difference is that the Poisson bracket is now replaced by the commutator bracket $[A, B] = AB - BA$ and the operators H, L_1 and L_2 are the obvious (formally self-adjoint) symmetry partial differential operators, built from the symmetry operators

$$J_1 = -z\partial_y \quad J_2 = z\partial_x \quad J_3 = x\partial_y - y\partial_x \tag{53}$$

where $z = \sqrt{1 - x^2 - y^2}$ and

$$H = J_1^2 + J_2^2 + J_3^2 + V(x, y) \quad L_h = \sum_{k,j=1}^2 \partial_k(a_{(h)}^{kj})\partial_j + W_{(h)}(x, y) \quad h = 1, 2. \tag{54}$$

The Hamilton–Jacobi equation is replaced by the Schrödinger equation

$$H\Psi = E\Psi. \tag{55}$$

Just as for the Hamilton–Jacobi case, if we have another constant of motion (symmetry operator) L_3 associated with a maximal potential, then L_3 must be a linear combination of H, L_1, L_2 . Indeed, if L_3 is in self-adjoint form, then the conditions that $[H, L_3] = 0$ are identical with (16), (where we replace $J_h J_k$ by $\frac{1}{2}\{J_h, J_k\}$) and (17), (18). Thus, if L_3 is not a linear combination of the basis functions, then the potential V must satisfy an equation (10) that is linearly independent of the equations associated with L_1, L_2 . This means an additional constraint on the solution space and that V can depend on at most two parameters, which is a contradiction.

Furthermore the proof of the existence of quadratic algebra relations at the end of section 2 goes through almost unchanged for the operator case: $[L_1, L_2]^2 = R^2$ and $[L_1, R], [L_2, R]$ can be expressed as (symmetric) polynomials in the operators L_1, L_2, H . To make the prior construction go through, one need only note that, since R^2 is a formally self-adjoint sixth-order differential symmetry operator, the fifth-order terms are fixed linear functions of the sixth-order terms. The expressions $\{A, B\} = AB + BA$ and $\{A, B, C\} = ABC + CAB + BCA$ are operator symmetrizers. The explicit relations are the following.

Case 1a.

$$\begin{aligned}
 [L_i, R] &= 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k) \\
 R^2 &= \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \\
 &\quad + \frac{52}{3}(\{L_1, L_2\} + \{L_1, L_3\} + \{L_2, L_3\}) + \frac{1}{3}(16 + 176a_1)L_1 \\
 &\quad + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) \\
 &\quad + 48(a_1a_2 + a_1a_3 + a_2a_3) + 64a_1a_2a_3
 \end{aligned} \tag{56}$$

where $a_1 = \gamma, a_2 = \alpha, a_3 = \beta, L_1 + L_2 + L_3 + \sum a_h = H$ and i, j, k are chosen such that $\epsilon_{ijk} = 1$, where ϵ is the purely skew-symmetric tensor.

Case 1b.

$$\begin{aligned}
 [R, L_2] &= 8L_2^2 + 16\gamma H + 8\beta L_2 + 16\gamma L_1 - (8\gamma - 16\alpha\gamma) \\
 [R, L_1] &= -8\{L_2, L_1\} - 8\beta H - 16L_2 - 8\beta L_1 + 8\beta(\alpha - 1) \\
 R^2 &= -\frac{8}{3}\{L_2, L_2, L_1\} - 16\gamma H^2 - \frac{176}{3}L_2^2 - 16\gamma L_1^2 - 16\beta HL_2 - 32\gamma HL_1 \\
 &\quad - 8\beta\{L_2, L_1\} - (-16\alpha\beta + \frac{176}{3}\beta)L_2 + (\frac{176}{3}\gamma - 32\alpha\gamma)L_1 \\
 &\quad + (32\alpha\gamma + \frac{32}{3}\gamma)H + (\frac{32}{3}\gamma + 16\beta^2\alpha + \frac{32}{3}\alpha\gamma - 16\alpha^2\gamma - 12\beta^2).
 \end{aligned} \tag{57}$$

Case 1c.

$$\begin{aligned}
 [R, L_2] &= 2L_0^2 - \frac{\alpha}{4}H + \alpha L_1 + \frac{1}{8}\alpha \\
 [R, L_1] &= -2\{L_2, L_1\} - L_2 \\
 R^2 &= -\frac{2}{3}\{L_2, L_2, L_1\} - \frac{11}{3}L_2^2 - \alpha L_1^2 + \alpha HL_1 + \frac{11}{12}\alpha L_1 - \frac{1}{12}\alpha H + \frac{\alpha}{4}\left(\beta^2 + \frac{1}{6}\right).
 \end{aligned} \tag{58}$$

Case 1d.

$$\begin{aligned}
 [R, L_1] &= -2\{L_1, L_2\} + 2\alpha L_1 - (2\gamma + 2\beta + 1)L_2 + (2\alpha\beta + \frac{1}{2}\alpha) \\
 [R, L_2] &= 6L_1^2 + 2L_2^2 - 4HL_1 + \frac{1}{2}H + (2\beta + 2\gamma - 3)L_1 - 2\alpha L_2 - (\beta + \gamma + 2\beta\gamma) \\
 R^2 &= -4L_1^3 - \frac{2}{3}\{L_2, L_2, L_1\} + 4L_1^2H + (11 - 2\beta - 2\gamma)L_1^2 \\
 &\quad + (-\frac{11}{3} - 2\beta - 2\gamma)L_2^2 - \frac{11}{3}HL_1 + \alpha\{L_1, L_2\} \\
 &\quad + 2(\frac{22}{3}(\beta + \gamma) + 4\beta\gamma)L_1 + (\frac{11}{3}\alpha + 4\beta\alpha)L_2 + (-\frac{1}{6} - \frac{3}{2}\gamma - 4\beta\gamma - \frac{3}{2}\beta)H \\
 &\quad + (-\frac{1}{6}\beta - \frac{1}{6}\gamma + \frac{3}{4}\beta^2 + \frac{3}{4}\gamma^2 - 2\beta\alpha^2 + 2\beta^2\gamma + \frac{13}{6}\beta\gamma - \frac{3}{4}\alpha^2 + 2\beta\gamma^2).
 \end{aligned} \tag{59}$$

(60)

Case 2.

$$\begin{aligned}
[R, L_2] &= -8c^4H^2 - 8(2c^4 - 1)L_2^2 + 16c^2(2c^2 - 1)HL_2 + 8\{L_2, L_1\} \\
&\quad + 16c^2(2c^2 - 1)H + 8(4\gamma c^2 + 1 - 2c^4)L_2 + 16c^4L_1 + 8[(\alpha - \beta)^2 + 4\gamma]c^4, \\
[R, L_1] &= -8(c^2 - 1)(5c^2 + 1)c^2H^2 - 24c^2(c^4 - 1)L_2^2 - 8c^2L_1^2 \\
&\quad + 32c^2(c^2 - 1)(2c^2 + 1)HL_2 - 16c^2(2c^2 - 1)HL_1 + 8(2c^2 - 1)\{L_2, L_1\} \\
&\quad + 16c^2(2c^6 - c^4 - 4c^2\gamma - c^2 + 4\gamma + 1)H \\
&\quad + 16(-c^8 + c^4 + 4c^4\gamma - 4\gamma - 1)L_2 \\
&\quad + 8c^2(-4\gamma + 2c^4 - 1)L_1 + 8c^2[(c^2(\alpha - \beta))^2 - (\alpha + \beta)^2 + 4c^4\gamma] \\
R^2 &= 32c^4(c^2 - 1)H^3 - 16c^2(c^4 - 1)L_2^3 - \frac{16}{3}c^2\{L_1, L_1, L_2\} \\
&\quad + 32c^2(c^2 - 1)(2c^2 + 1)L_2^2H + \frac{8}{3}(2c^2 - 1)\{L_2, L_2, L_1\} \\
&\quad - 16(c^2 - 1)(5c^2 + 1)c^2H^2L_2 + 16c^4H^2L_1 \\
&\quad - 16c^2(2c^2 - 1)\{L_2, L_1\} - \frac{176}{3}c^4(2c^2 - 1)^2H^2 \\
&\quad + (64\gamma(c^4 - 1) - \frac{176}{3}(c^8 + c^4 - 1))L_2^2 - \frac{176}{3}c^4L_1^2 \\
&\quad + (-128\gamma c^2(c^2 - 1) + \frac{352}{3}c^2(2c^6 - c^4 - c^2 - 1))HL_2 \\
&\quad + (-32\gamma c^2 + \frac{88}{3}c^2(2c^4 - 1))\{L_2, L_1\} + (-16c^4(\beta - \alpha)^2 - \frac{704}{3}c^2\gamma)L_1 \\
&\quad + (16c^2[(c^2(\alpha - \beta))^2 - (\alpha + \beta)^2] + \frac{704}{6}c^2\gamma)L_2 \\
&\quad + (-32c^4(\beta - \alpha)(c^2(\beta - \alpha) + \beta + \alpha) + \frac{32}{3}c^4 - \frac{128}{3}(11c^2 - 5)c^4\gamma)H \\
&\quad - 64c^4\gamma(\beta - \alpha)^2 + \frac{32}{3}c^4\alpha^2 - 256\gamma^2c^4 + \frac{64}{3}c^4\gamma + \frac{32}{3}c^4\beta^2.
\end{aligned} \tag{61}$$

Case 3.

$$\begin{aligned}
[L_2, R] &= -6L_1^2 - 8\gamma H + \beta H + 8\alpha L_1 - 2(\alpha^2 + 3\gamma) \\
[L_1, R] &= -16\gamma L_2 - 2\beta L_1 + 2\alpha\beta \\
R^2 &= -4L_1^3 + 16\gamma L_2^2 + 8\alpha L_1^2 - 16\gamma HL_1 + 2\beta\{L_2, L_1\} + 4\alpha\beta L_2 \\
&\quad - (44\gamma + 4\alpha^2)L_1 - \beta^2 H + (-\frac{3}{4}\beta^2 + 32\gamma\alpha).
\end{aligned} \tag{62}$$

$$\tag{63}$$

We note that the quadratic relations in the quantum case provide useful information relating the special functions that occur as (separable) eigenfunctions for each superintegrable case [20–22]. For other applications of superintegrability on the real sphere or the real hyperboloid see [20–25].

4. Conclusions

In this paper we have used the concept of a ‘nondegenerate potential’ to add structure to the study of superintegrable classical and quantum mechanical systems on the complex 2-sphere. We have shown how to classify all such systems in a straightforward manner, so that gaps can be avoided. In particular, potentials V_5 and V_6 in our table appear to be new. Furthermore, we have shown the following.

- (1) Each system is associated with a pair of constants of motion in the classical case, and a pair of symmetry operators in the quantum case, that generate a quadratic algebra.
- (2) There is a one-to-one correspondence between second-order superintegrable systems and free-field symmetry operators that generate quadratic algebras.
- (3) Second-order superintegrability implies multiseparability, i.e. separability in more than one coordinate system.

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Appendix

As is well known [17, 26], there are essentially five coordinate systems on the complex 2-sphere in which the free-particle Hamilton–Jacobi equation separates: spherical, elliptic, horospherical and degenerate elliptic of the first and second types. We describe these coordinate systems and their corresponding free-particle constants of motion L . (We adopt the basis $J_1 = -zp_y$, $J_2 = zp_x$, $J_3 = xp_y - yp_x$, for the Lie algebra $so(3, C)$, where $z = \sqrt{1 - x^2 - y^2}$.) The systems are the following.

Spherical coordinates

$$\begin{aligned} x &= \sin \theta \cos \varphi & y &= \sin \theta \sin \varphi \\ z &= \cos \theta & L &= J_3^2. \end{aligned} \quad (64)$$

Elliptic coordinates

$$\begin{aligned} x^2 &= \frac{(ru - 1)(rv - 1)}{1 - r} & y^2 &= \frac{r(u - 1)(v - 1)}{1 - r} \\ z^2 &= ruv & L &= J_1^2 + rJ_2^2. \end{aligned} \quad (65)$$

Horospherical coordinates

$$\begin{aligned} x &= \frac{i}{2} \left(v + \frac{u^2 - 1}{v} \right) & y &= \frac{1}{2} \left(v + \frac{u^2 + 1}{v} \right) \\ z &= \frac{iu}{v} & L &= (J_1 + iJ_2)^2. \end{aligned} \quad (66)$$

Degenerate elliptic coordinates of type 1

$$\begin{aligned} x + iy &= \frac{4cuv}{(u^2 + 1)(v^2 + 1)} & x - iy &= \frac{(u^2v^2 + 1)(u^2 + v^2)}{cuv(u^2 + 1)(v^2 + 1)} \\ z &= \frac{(u^2 - 1)(v^2 - 1)}{(u^2 + 1)(v^2 + 1)} & L &= (J_1 + iJ_2)^2 - c^2J_3^2. \end{aligned} \quad (67)$$

Degenerate elliptic coordinates of type 2

$$\begin{aligned} x + iy &= -iuv & x - iy &= \frac{i}{4} \frac{(u^2 + v^2)^2}{u^3v^3} \\ z &= \frac{i}{2} \frac{u^2 - v^2}{uv} & L &= J_3(J_1 - iJ_2). \end{aligned} \quad (68)$$

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